# Solução via autovalores e autovetores

## aula 4 - 2 spp

$$x_{n+1} = a_{11} x_n + a_{12} y_n$$

$$y_{n+1} = a_{21} x_n + a_{22} y_n$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$$v_{n+1} = \underbrace{A} v_n$$

$$x_{n+1} = \beta x_n - \gamma x_{n-1}$$

$\beta$ = traço de A

$\gamma$ = det da A

Traço de A: $a_{11} + a_{22}$

Det de A: $a_{11}a_{22} - a_{12}a_{21}$

$\det(A) \neq 0$ $\left\{ \begin{array}{l} \text{2 vetores especiais} \\ \text{(autovetores)} \end{array} \right.$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

autovalores

$$A v_+ = \lambda_+ v_+ \qquad A v_- = \lambda_- v_- \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \lambda I \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}}_{v} = 0$$

trivial: $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$M \quad v = 0$$

Soluții non-trivial: $\det(M) = 0$

$$(a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21} = 0$$

$$a_{11}a_{22} - \lambda a_{11} - \lambda a_{22} + \lambda^2 - a_{12}a_{21} = 0$$

Eq caract: $\lambda^2 - \lambda \underbrace{(a_{11}+a_{22})}_{tr(A)} + \underbrace{(a_{11}a_{22} - a_{12}a_{21})}_{\det(A)} = 0$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{cases} a_{11} v_1 + a_{12} v_2 = \lambda v_1 \\ a_{21} v_1 + a_{22} v_2 = \lambda v_2 \end{cases}$$

$$\longrightarrow \quad a_{11} v_1 + a_{12} v_2 - \lambda v_1 = 0$$

$$(a_{11} - \lambda) v_1 = -a_{12} v_2$$

$$v_1 = \frac{-a_{12} v_2}{a_{11} - \lambda}$$

$$a_{21} \left( \frac{-a_{12} v_2}{a_{11} - \lambda} \right) + a_{22} v_2 - \lambda v_2 = 0$$

$$\left( \frac{-a_{21} a_{12} + a_{22}(a_{11} - \lambda) - \lambda (a_{11} - \lambda)}{a_{11} - \lambda} \right) v_2 = 0$$

$$-a_{21} a_{12} + a_{22} a_{11} - a_{22} \lambda - \lambda a_{11} + \lambda^2 = 0$$

$$-a_{12}a_{21} + a_{22}a_{11} - \lambda a_{22} - \lambda a_{11} + \lambda^2 = 0$$

$$\underbrace{\lambda^2} - \lambda\underbrace{(a_{11}+a_{22})}_{\mathrm{tr}A=\beta} + \underbrace{(a_{22}a_{11} - a_{12}a_{21})}_{\det A = \gamma} = 0$$

$$x_{n+1} = \beta x_n - \gamma \rightsquigarrow x_{n+1} - \beta x_n + \gamma = 0$$

$$\underbrace{R^2} - \beta R + \gamma = 0$$

$$\lambda_\pm = \frac{\beta}{2} \pm \frac{1}{2}\sqrt{\beta^2 - 4\gamma}$$

$$(a_{11}-\lambda)\underbrace{v_1}_{-a_{12}} = -a_{12}\underbrace{v_2}_{(a_{11}-\lambda)}$$

$$v = \begin{pmatrix} -a_{12} \\ a_{11}-\lambda \end{pmatrix}$$

$$(a_{11}-\lambda)(-a_{12}) = (-a_{12})(a_{11}-\lambda)$$

$$\underbrace{\lambda_\pm = \frac{\beta}{2} \pm \frac{1}{2}\sqrt{\beta^2-4\gamma}}$$

$$\begin{cases} v_+ = \begin{pmatrix} -a_{12} \\ a_{11}-\lambda_+ \end{pmatrix} \\ v_- = \begin{pmatrix} -a_{12} \\ a_{11}-\lambda_- \end{pmatrix} \end{cases}$$

$$v_n = \underbrace{C_+}\lambda_+^n v_+ + \underbrace{C_-}\lambda_-^n v_-$$

$$Av = \lambda v$$

$$\boxed{V_n = C_+ \lambda_+^n V_+ + C_- \lambda_-^n V_-}$$

$$Av = \lambda v$$

$$v_{n+1} = A v_n$$

$$\underbrace{A V_n} = C_+ \lambda_+^n A v_+ + C_- \lambda_-^n A v_-$$

$$v_{n+1} = C_+ \lambda_+^n \lambda_+^1 v_+ + C_- \lambda_-^n \lambda_-^1 v_-$$

$$= \underbrace{C_+} \lambda_+^{n+1} v_+ + \underbrace{C_-} \lambda_-^{n+1} v_-$$

cond. inicial

$$v_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$V_0 = C_+ V_+ + C_- V_- \longrightarrow \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = C_+ \begin{pmatrix} v_{1+} \\ v_{2+} \end{pmatrix} + C_- \begin{pmatrix} v_{1-} \\ v_{2-} \end{pmatrix}$$

Exemplo:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}}_{A} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$v_1 + v_2 = \lambda v_1 \rightsquigarrow v_2 = \lambda v_1 - v_1$$

$$(\lambda - 1) v_1$$

$$-2v_1 + 4v_2 = \lambda v_2 \qquad v_2 = (\lambda - 1) v_1$$

$$-2v_1 + 4(\lambda - 1) v_1 = \lambda (\lambda - 1) v_1$$

$$-2v_1 + 4\lambda v_1 - 4 v_1 = \lambda^2 v_1 - \lambda v_1$$

$$(\lambda^2 - \lambda - 4\lambda + 2 + 4) v_1 = 0$$

$$(\lambda^2 - 5\lambda + 6)\, v_1 = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$\begin{cases} \lambda = \dfrac{5}{2} \pm \dfrac{1}{2}\sqrt{\overbrace{25-24}^{=1}} \\ \underline{\lambda_1 = 3} \qquad \underline{\lambda_2 = 2} \end{cases}$$

$$v_2 = (\lambda - 1)\, v_1$$

$$\lambda_1 = 3 \quad v_2 = (3-1)\, v_1 = 2v_1 \qquad \begin{pmatrix} v_1 \\ 2v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\lambda_2 = 2 \quad v_2 = (2-1)\, v_1 = v_1 \qquad \begin{pmatrix} v_1 \\ v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$V_n = C_1 (3)^n \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 (2)^n \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$V_n = C_1 (3)^n \begin{pmatrix}1\\2\end{pmatrix} + C_2 (2)^n \begin{pmatrix}1\\1\end{pmatrix} \qquad V_0 = \begin{pmatrix}10\\5\end{pmatrix}$$

2ª ordem

2 condições iniciais

$x_0$

$x_1$

$$\begin{pmatrix}10\\5\end{pmatrix} = C_1 \begin{pmatrix}1\\2\end{pmatrix} + C_2 \begin{pmatrix}1\\1\end{pmatrix}$$

$$10 = C_1 + C_2 \quad \dashrightarrow \quad 10 = C_1 + 5 - 2C_1 \Rightarrow 10 = 5 - C_1$$

$$-C_1 = 5$$

$$5 = 2C_1 + C_2 \longrightarrow C_2 = 5 - 2C_1 \qquad C_1 = -5$$

$$C_2 = 5 - 2\cdot(-5) = 5 + 10 = 15$$

$$\boxed{V_n = (-5)(3)^n \begin{pmatrix}1\\2\end{pmatrix} + 15 (2)^n \begin{pmatrix}1\\1\end{pmatrix}}$$

DESAFIO:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}}_{A} \begin{pmatrix} x_n \\ y_n \end{pmatrix} \qquad V_0 = \begin{pmatrix} 10 \\ 20 \end{pmatrix}$$

Encontrem os autovalores de A resultados qualitativo